

A Goldstone Theorem in Thermal Relativistic Quantum Field Theory

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We prove a Goldstone Theorem in thermal relativistic quantum field theory, which relates spontaneous symmetry breaking to the rate of space-like decay of the two-point function. The critical rate of fall-off coincides with that of the massless free scalar field theory. Related results and open problems are briefly discussed.

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I. INTRODUCTION AND SUMMARY

Thermal quantum field theory (*tgft*) has received considerable attention recently, both from the conceptual and the constructive point of view (see [1] for a review and references). Its range of applications extends from heavy ion collisions and cosmology at early stages (see [2] for a review) to the present day hot big-bang model in cosmology [3], with obvious potential relevance to the dark energy problem [4], which, however, remains to be explored.

In the present paper, we study the spontaneous symmetry breaking (*ssb*) of continuous (internal) symmetries in relativistic thermal quantum field theory, and prove a version of Goldstone's theorem (see, e.g., [5] for a review and references) — Theorem III.3 of Section III — which relates *ssb* to the asymptotic decay of (truncated) correlation functions for large space-like distances. In this respect the theorem follows the lines of [6] and [7], the latter having been proved to be an optimal version, generalizing the well-known Mermin-Wagner theorem of quantum statistical mechanics [8]. If, however, one endeavors to understand the concept and structure of particles in *tgft*, large time-like distances necessarily come into play, and in this connection the Goldstone-type theorem of Bros and Buchholz [9] is more natural (see Remark III.1 of Section III and Section IV — discussion and outlook).

The main advantage of our approach lies in the possibility of a sharp distinction between massive and zero-mass theories in terms of their correlation functions' rate of space-like decay (Conjecture III.4 of Section III): if such is true, a theorem of the same form as the vacuum ($T=0$, zero density) version [10, 11] follows (Corollary III.5) and Theorem III.3 turns out to be optimal as in the quantum statistical mechanical case.

Our proof of Theorem III.3 generalises the method

used in [6, 7], which was based on the Bogoliubov inequality (see [12] and the references given there), in an essential way: firstly, the treatment of the middle term in that inequality (see (82) of Appendix A) relies on local current conservation, Einstein causality and the definition (12) of the global charge, in a manner reminiscent of [13]; secondly, unlike in [6, 7], we employ a form of the Bogoliubov inequality which was proved to follow from the KMS condition for infinite systems by Garrison and Wong [14] in a C^* -algebraic framework. This naturally takes into account the singular nature of the quantum fields, which reflects itself in the necessity of choosing adequate test functions. For other related derivations, see [15] and [16], and [17], Vol. II, pg. 334.

We now briefly describe the organisation of the paper. In Section II we introduce the framework, which is that of C^* -dynamical systems (see e.g. [25]) and formulate our assumptions, together with some auxiliary lemmas. In Section III we prove our main result (Theorem III.3), followed by Conjecture III.4 and Corollary III.5 referred to before. The connection to W^* -dynamical systems and the spectral properties of the Liouvillian is also discussed there. Section IV is reserved for a discussion and outlook, in particular the relation to other approaches and open problems. In Appendix A we state Bogoliubov's inequality, with some additions needed in the main text. In Appendix B we state, for the reader's convenience, the theorem on the partition of unity used in the main text.

II. FRAMEWORK AND ASSUMPTION

We work in the framework of C^* -dynamical systems, consisting of a pair (\mathcal{A}, τ) , where \mathcal{A} is a C^* -algebra with unit and $\{\tau_t\}_{t \in \mathbb{R}}$ is a one-parameter group of norm continuous (time translation) automorphisms of \mathcal{A} (see, e.g., [25]). Since the time-translation automorphisms are not norm continuous on the Weyl algebra (see [17], Vol. 2, Theorem 5.2.8), we adopt Haag's construction (see [18], pp. 129 et seq.), which leads to the following structure (see [18], III.3.3., pg. 141) - \mathcal{O} denotes a finite, con-

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tractible, open region in Minkowski space:

- (i) a net of C^* -algebras with common unit

$$\mathcal{O} \rightarrow \mathcal{A}_S(\mathcal{O}) \quad (1)$$

with the total C^* -algebra (the C^* -inductive limit [19]) \mathcal{A}_S :

$$\mathcal{A}_S = \overline{\mathcal{A}_L}, \quad \text{with } \mathcal{A}_L = \bigcup_{\mathcal{O}} \mathcal{A}_S(\mathcal{O}), \quad (2)$$

where the bar denotes the completion in the norm topology. We call \mathcal{A}_L the (strictly) local algebra. The action of the time automorphism $\{\tau_t\}_{t \in \mathbb{R}}$ on \mathcal{A}_S is t -continuous in the norm topology (in [18], this is required for the space-time automorphisms, but we do not need the smoothness with respect to spatial translations);

- (ii) a set \mathcal{S} of physical (i.e., locally normal [18, 19]) states over \mathcal{A}_S and the complex linear span of \mathcal{S} , denoted by Σ ;
- (iii) the dual of Σ is a net of W^* -algebras with common unit

$$\mathcal{O} \mapsto \mathcal{R}(\mathcal{O}) = \Sigma(\mathcal{O})^*. \quad (3)$$

$\mathcal{R}(\mathcal{O})$ is closed in the weak topology induced by Σ and $\mathcal{A}_S(\mathcal{O})$ is weakly dense in $\mathcal{R}(\mathcal{O})$.

- (iv) Local commutativity: if the regions \mathcal{O}_1 and \mathcal{O}_2 are totally space-like to one another, then

$$[A, B] = 0 \quad \forall A \in \mathcal{A}(\mathcal{O}_1), \quad \forall B \in \mathcal{A}(\mathcal{O}_2). \quad (4)$$

We must also assume certain global properties on the particular state $\omega \in \mathcal{S}$ we shall work with. The basic property of thermal states is the KMS condition (see [17], Vol. 2):

Definition II.1 A state ω ($= \omega_\beta$) over \mathcal{A} ($= \mathcal{A}_S$) is called a KMS state for some $\beta > 0$, if for all $A, B \in \mathcal{A}$ there exists a function $F_{A,B}$, which is continuous in the strip $0 \leq \Im z \leq \beta$ and analytic and bounded in the open strip $0 < \Im z < \beta$, with boundary values given by

$$F_{A,B}(t) = \omega(A\tau_t(B)) \quad (5)$$

and

$$F_{A,B}(t + i\beta) = \omega(\tau_t(B)A) \quad (6)$$

for all $t \in \mathbb{R}$.

We further assume that

- A1** ω is a factor (primary) state over \mathcal{A}_S ;
- A2** ω satisfies the KMS condition.

From **A2** it follows that ω is invariant under time translations, but we also need that

A3 ω is invariant under space translations.

By **A2**, **A3** and the GNS construction there exists a representation π_ω of \mathcal{A}_S on a Hilbert space \mathcal{H}_ω , with self-adjoint space-time translation generators $(L_\omega, \vec{P}_\omega)$ and cyclic vector Ω_ω such that

$$L_\omega \Omega_\omega = 0, \quad (7)$$

and

$$\vec{P}_\omega \Omega_\omega = \vec{0}. \quad (8)$$

As occurs with W^* -dynamical systems, L_ω is not bounded below, see (74) et seq.. Of primary concern to us will be continuous internal symmetries generated by a local current $J_\mu(x)$ on which we make the same assumptions as in [10] (see p. 110), headed there under local current conservation. Before stating them we remark that in the following, when $A \in \mathcal{A}_L$ occurs in connection with the representation π_ω , it is understood as $\pi_\omega(A)$. The assumptions are: there exist for every test-function $f \in \mathcal{D} = C_0^\infty(\mathbb{R}^{s+1})$ a set of $(s+1)$ unbounded self-adjoint operators $J_\mu(f)$, with the properties

A4 Ω_ω is in the domain of $J_\mu(f)$ for all $f \in \mathcal{D}$;

A5 $T(a)J_\mu(f)T(a)^{-1} = J_\mu(f_a)$ where $f_a(x) = f(x-a)$;

A6 $\sum_{\mu=0}^s J_\mu(\frac{\partial f}{\partial x_\mu}) = 0$;

A7 (a) $(\Omega_\omega, [J_\mu(f), A]\Omega_\omega) = 0$ for $A \in \mathcal{A}_S(\mathcal{O})$, if the support of f is totally space-like to \mathcal{O} ;

(b) $(\Omega_\omega, [J_0(f), \vec{J}(g)]\Omega_\omega) = \vec{0}$, if the supports of f and g are space-like to one another;

A8 for all $f \in \mathcal{D}$, the charge operator $J_0(f)$ is affiliated to $\mathcal{R}(\mathcal{O})$.

In **A7** (a) the natural definition

$$(\Omega_\omega, [J_\mu(f), A]\Omega_\omega) = (J_\mu(f)\Omega_\omega, A\Omega_\omega) - (A^*\Omega_\omega, J_\mu(f)\Omega_\omega) \quad (9)$$

is adopted. By **A4** and (9), the term

$$(\Omega_\omega, [J_0(f), \vec{J}(g)]\Omega_\omega) \quad (10)$$

is well-defined for all $f, g \in \mathcal{D}$. **A7**(b) follows from the assumption e.) of [10] (see pg. 110) that $(\Omega_\omega, [J_0(f), \vec{J}(g)]\Omega_\omega)$ is a tempered distribution, but we only need **A7**(b).

Assumption **A8** had to be imposed on $J_\mu(f)$, because the KMS condition (5) a priori holds only for $A, B \in \mathcal{A}$. Recall that $\mathcal{R}(\mathcal{O})$ is the von Neumann algebra defined in (3), and the concept of affiliation is defined in [17] (see Vol. 1, Definition 2.5.7, pg. 87). By self-adjointness

of $J_0(f)$ and [17], Lemma 2.5.8 (see Vol. 1, pg. 87) the spectral projections $E(\lambda)$ of $J_0(f)$ lie in $\mathcal{R}(\mathcal{O})$. Note that it is too much to require that they lie in $\mathcal{A}_S(\mathcal{O})$: bounded functions of the fields are expected to belong *only* to the weak closure of the Weyl algebra, and are thus not smooth elements.

We have now completed our list of assumptions, and turn to our criterion of existence of *ssb*, which is the same as the one adopted in [10]. One might expect that the limit $V \rightarrow \infty$ of the local integrated current density

$$\int_V d^s \vec{x} J_0(x_0, \vec{x}) \quad (11)$$

defines a global charge operator, which serves as the generator of the internal symmetry considered. However, the limit $V \rightarrow \infty$ of (11) does not exist due to vacuum fluctuation occurring all over space, by translation invariance: this is, in fact, as remarked in [13], the source of *ssb*. We therefore define, as in [10], the charge operator corresponding to J_μ as a suitable limit of the operators

$$J_0(f_d \otimes g_R) := \int d^{s+1}x f_d(x_0) g\left(\frac{\vec{x}}{R}\right) J_0(x) \quad (12)$$

as $R \rightarrow \infty$, where

$$g \in \mathcal{D}_s := C_0^\infty(\mathbb{R}^s), \quad (13)$$

$$f_d \in \mathcal{D} := C_0^\infty(\mathbb{R}), \quad (14)$$

$$\int dx_0 f_d(x_0) = 1, \quad (15)$$

$$f_d(x_0) = 0 \quad \text{if} \quad |x_0| \geq d, \quad (16)$$

$$g(\vec{x}) = 1 \quad \text{if} \quad |\vec{x}| \leq 1, \quad (17)$$

$$g(\vec{x}) = 0 \quad \text{if} \quad |\vec{x}| > 1 + \delta, \quad 0 < \delta < 1, \quad (18)$$

with $\vec{J}(f \otimes g_R)$ defined similarly. For other choices, see ref. [35]. The symmetry is characterized by the following property (see [10], p. 111): there exists a one-parameter group of automorphisms $A \mapsto A^\lambda$ of \mathcal{A}_S , strongly continuous with respect to λ , such that:

(a) if $\mathcal{O}_L = \{x \in \mathbb{R}^{s+1} \mid |\vec{x}| + |x_0| < L\}$, $L > 0$, then

$$A_L \in \mathcal{A}_S(\mathcal{O}_L) \quad \text{implies} \quad A_L^\lambda \in \mathcal{A}_S(\mathcal{O}_L); \quad (19)$$

(b) if f_d, g satisfy (13)–(18) and $J_0(f_d \otimes g_R)$ is defined by (12), then

$$\begin{aligned} & \left. \frac{d}{d\lambda} (\Omega_\omega, A^\lambda \Omega_\omega) \right|_{\lambda=0} \\ &= i \lim_{R \rightarrow \infty} (\Omega_\omega, [J_0(f_d \otimes g_R), A] \Omega_\omega). \end{aligned} \quad (20)$$

Lemma II.2 *About the limit on the r.h.s. of (20) we have*

$$\begin{aligned} & \lim_{R \rightarrow \infty} (\Omega_\omega, [J_0(f_d \otimes g_R), A_L] \Omega_\omega) \\ &= (\Omega_\omega, [J_0(f_d \otimes g_{R_1}), A_L] \Omega_\omega) \end{aligned} \quad (21)$$

if

$$R_1 \geq L + d + 1. \quad (22)$$

The r.h.s. in (21) is independent of the functions f_d and g , as long as they satisfy (13)–(18).

Proof This follows from local commutativity **A7**(a), see Lemma 1, pg. 112, of [10]. ■

CRITERION. Given (20), we adopt as in [10] as our criterion for spontaneous breakdown of the symmetry (*ssb*) associated to the one-parameter group of automorphisms of \mathcal{A}_S :

$$\lim_{R \rightarrow \infty} (\Omega_\omega, [J_0(f_d \otimes g_R), A_0] \Omega_\omega) = c \neq 0 \quad (23)$$

for some $A_0 \in \mathcal{A}_L$.

In this paper we shall assume that (23) holds, and derive some constraints from it.

There are two preliminary steps, which we shall prove in this section: firstly, using time translation invariance, which follows from **A2**, we show that we may replace $J_0(f_d \otimes g_R)$ in (23) by a smoothened version, essential to the application of Bogoliubov's inequality in Section III. Secondly, using **A3** we show that A_0 may be replaced by an average over space-translations, which is essential to relate (23) to the rate of space-like decay of correlations.

Lemma II.3 *Assume ssb takes place, in the sense that (23) holds. Then there exists $h \in \mathcal{D}(\mathbb{R})$ such that (23) also holds (with a different $c \neq 0$) for the observable*

$$\tilde{A}_0 := \int dt h(t) \tau_t(A_0). \quad (24)$$

Proof By Lemma II.2

$$\begin{aligned} & \lim_{R \rightarrow \infty} (\Omega_\omega, [J_0(f_d \otimes g_R), A_0] \Omega_\omega) \\ &= (\Omega_\omega, [J_0(f_d \otimes g_{R_1}), A_0] \Omega_\omega) \end{aligned} \quad (25)$$

if

$$R_1 = L + d + 1. \quad (26)$$

Now write (25) as

$$(J_0(f_d \otimes g_{R_1}) \Omega_\omega, A_0 \Omega_\omega) - (A_0^* \Omega_\omega, J_0(f_d \otimes g_{R_1}) \Omega_\omega). \quad (27)$$

By (26), (27) and **A4**, and the norm-continuity of the time evolution τ_t as assumed in (i), given $\epsilon > 0$, we may choose $h_\epsilon \in \mathcal{D}$ such that

$$|(\Omega_\omega, [J_0(f_d \otimes g_R), A_0]\Omega_\omega) - (\Omega_\omega, [J_0(f_d \otimes g_{R_1}), \tilde{A}_0]\Omega_\omega)| < \epsilon. \quad (28)$$

See [20], Theorem 4.8, for the proof of (28). The identities (25) and (28) imply

$$\begin{aligned} & \lim_{R \rightarrow \infty} (\Omega_\omega, [J_0(f_d \otimes g_R), \tilde{A}_0]\Omega_\omega) \\ &= (\Omega_\omega, [J_0(f_d \otimes g_{R_1}), \tilde{A}_0]\Omega_\omega) = c_\epsilon \neq 0. \end{aligned} \quad (29)$$

■

In the following we omit the suffixes ϵ in h_ϵ and c_ϵ .

Lemma II.4 *Assume ssb takes place, in the sense that (23) holds. Then*

$$\begin{aligned} & \lim_{R \rightarrow \infty} (\Omega_\omega, [J_0(f_d \otimes g_R), \tilde{A}_0]\Omega_\omega) \\ &= (\Omega_\omega, [I_h(f_d \otimes g_{R_1}), A_0]\Omega_\omega) \\ &= c \neq 0, \end{aligned} \quad (30)$$

where

$$I_h(f_d \otimes g_R) := \int dt h(t) \tau_{-t}(J_0(f_d \otimes g_R)). \quad (31)$$

h is the function in Lemma II.3 and R_1 is given by (26).

Proof By **A4** and **A5**,

$$I_h(f_d \otimes g_R)\Omega_\omega = \int dt h(t) e^{-itL_\omega} J_0(f_d \otimes g_{R_1})\Omega_\omega, \quad (32)$$

since $t \mapsto e^{-itL_\omega} J_0(f_d \otimes g_{R_1})\Omega_\omega \in \mathcal{H}_\omega$ is continuous, (30) is meaningful and follows from (29) by time translation invariance of $\omega(\cdot) := (\Omega_\omega, \cdot \Omega_\omega)$. ■

Our last preliminary lemma makes use of **A2**:

Lemma II.5 *Assume ssb takes place, in the sense that (23) holds. Then, for any $R_0 \in \mathbb{R}$,*

$$\begin{aligned} & \lim_{R \rightarrow \infty} (\Omega_\omega, [J_0(f_d \otimes g_R), \tilde{A}_0]\Omega_\omega) \\ &= (\Omega_\omega, [I_h(f_d \otimes g_{\tilde{R}_0}), A_{R_0}]\Omega_\omega) = c \neq 0, \end{aligned} \quad (33)$$

where

$$\tilde{R}_0 := 2R_0 + L + d + 1 \quad (34)$$

and

$$A_{R_0} = \frac{1}{|L_{R_0}|} \int_{L_{R_0}} d^s \vec{x} (\sigma_{\vec{x}}(A_0) - \omega(A_0)). \quad (35)$$

Above, L_{R_0} is a s -dimensional region of volume

$$|L_{R_0}| = O(R_0^s), \quad (36)$$

and $\sigma_{\vec{x}}(A_0) \equiv \pi_\omega(\sigma_{\vec{x}}(A_0)) = e^{i\vec{P}\vec{x}} \pi_\omega(A_0) e^{-i\vec{P}\vec{x}}$.

Proof Applying Lemma II.4, (30), and space-translation invariance **A2**,

$$\begin{aligned} & \lim_{R \rightarrow \infty} (\Omega_\omega, [J_0(f_d \otimes g_R), \tilde{A}_0]\Omega_\omega) \\ &= (\Omega_\omega, [I_h(f_d \otimes g_{\tilde{R}}), \sigma_{\vec{x}}(A_0)]\Omega_\omega) \end{aligned} \quad (37)$$

for any $\vec{x} \in \mathbb{R}^s$, where, by **A5**,

$$g_{\tilde{R}}(\vec{y}) = g_R(\vec{y} - \vec{x}). \quad (38)$$

We assume that as a consequence of finite speed of light, the tiny support of h can be taken into account by slightly increasing d . By (37), (38) and Lemma II.2,

$$\begin{aligned} & \lim_{R \rightarrow \infty} (\Omega_\omega, [J_0(f_d \otimes g_R), \tilde{A}_0]\Omega_\omega) \\ &= (\Omega_\omega, [I_h(f_d \otimes g_{R_1}), \sigma_{\vec{x}}(A_0)]\Omega_\omega) \end{aligned} \quad (39)$$

as long as

$$R_1 \geq 2|\vec{x}| + L + d + 1 \quad (40)$$

(33) and (34) follow from (39), (40) and (35). ■

(33) is the starting point for proving our main results in the next section.

III. A GOLDSTONE THEOREM IN THERMAL FIELD THEORY

As a preliminary to our proof of the Goldstone theorem, we write the self-adjoint operator in (33):

$$J_0(f_d \otimes g_{\tilde{R}_0}) = \lim_{n \rightarrow \infty} \lim_m \sum_{j=1}^m \lambda'_j E(\lambda_{j-1}, \lambda_j] \quad (41)$$

where $\lambda'_j \in (\lambda_{j-1}, \lambda_j]$ is arbitrary,

$$-n = \lambda_0 < \lambda_1 < \dots < \lambda_m = n \quad (42)$$

and \lim_m is the limit when $\max\{|\lambda_j - \lambda_{j-1}| \mid j = 1, \dots, m\}$ tends to zero. This is the spectral theorem (see, e.g., [21], p. 342). We shall abbreviate the double limit in (41) by $n, m \rightarrow \infty$, and denote the finite sum $\sum_{j=1}^m \lambda'_j \cdot E(\lambda_{j-1}, \lambda_j]$ by $J_0^{n,m}(f_d \otimes g_{R_1})$. Above,

$$E(\lambda_{j-1}, \lambda_j] = E(-\infty, \lambda_j] - E(-\infty, \lambda_{j-1}] \quad (43)$$

are the (right-continuous) spectral projections associated to J_0 . By **A8** et seq., they belong to $\mathcal{R}(\mathcal{O})$, and thus $J_0^{n,m}(f_d \otimes g_{\tilde{R}_0}) \in \mathcal{R}(\mathcal{O})$ for any finite n, m . The limit (41) is understood to be acting on any vector in the domain of $J_0(f_d \otimes g_{\tilde{R}_1})$.

In correspondence to (33) and (41), we define

$$I_h^{n,m} := \int dt h(t) \tau_{-t}(J_0^{n,m}(f_d \otimes g_{\tilde{R}_0})). \quad (44)$$

By definition (44), $I_h^{n,m}$ is a smooth element of $\mathcal{R}(\mathcal{O})$ (ignoring once again the fact that one has to increase the

region \mathcal{O} by a small amount in order to accommodate for the spreading due to the convolution with h), and thus, by (iii), belongs to $\mathcal{A}_S(\mathcal{O})$. We now apply Corollary V.2 of Appendix A, with $C = I_h^{n,m}$, $A = A_{R_0} = A_{R_0}^*$ (this may be assumed without loss of generality, otherwise we may decompose $A = S + iT$ with $S = S^*$, $T = T^*$). This leads to imaginary and real parts of c in (33), and obtain from the Bogoliubov's inequality(86):

$$\begin{aligned} & \frac{1}{\beta} |\omega([I_h^{n,m}, A_{R_0}])|^2 \\ & \leq \omega \left(\left[I_h^{n,m}, i \left(\frac{d}{dt} \tau_t(I_h^{n,m}) \right) \Big|_{t=0} \right] \right) \omega(A_{R_0}^2). \end{aligned} \quad (45)$$

Lemma III.1 *Under the assumptions made,*

$$\begin{aligned} & \lim_{n,m \rightarrow \infty} \omega([I_h^{n,m}, A_{R_0}]) \\ & = \omega \left(\left[I_h(f_d \otimes g_{\tilde{R}_0}), A_{R_0} \right] \right), \end{aligned} \quad (46)$$

and

$$\begin{aligned} & \lim_{n,m \rightarrow \infty} \omega \left(\left[I_h^{n,m}, i \left(\frac{d}{dt} \tau_t(I_h^{n,m}) \right)^* \Big|_{t=0} \right] \right) \\ & = i\omega \left(\left[I_h(f_d \otimes g_{\tilde{R}_0}), \int dt h'(t) J_0(f_d(\cdot - t)) \otimes g_{\tilde{R}_0} \right] \right). \end{aligned} \quad (47)$$

Proof By **A4**, **A5** and the spectral theorem,

$$\begin{aligned} & e^{itL\omega} J_0^{n,m}(f_d \otimes g_{\tilde{R}_0}) \Omega_\beta \\ & \xrightarrow{n,m \rightarrow \infty} J_0(f_d(\cdot - t)) \otimes g_{\tilde{R}_0} \Omega_\beta \end{aligned} \quad (48)$$

uniformly in $t \in \mathbb{R}$. Hence, by (44),

$$\begin{aligned} & I_h^{n,m} \Omega_\omega = \int dt h(t) e^{itL\omega} J_0^{n,m}(f_d \otimes g_{\tilde{R}_0}) \Omega_\omega \\ & \xrightarrow{n,m \rightarrow \infty} \int dt h(t) J_0(f_d(\cdot - t)) \otimes g_{\tilde{R}_0} \Omega_\omega \end{aligned} \quad (49)$$

from which (46) follows. Again by (44),

$$\left(\frac{d}{dt} \tau_t(I_h^{n,m}) \right) \Big|_{t=0} = \int dt h'(t) \tau_t \left(J_0^{n,m}(f_d \otimes g_{\tilde{R}_0}) \right). \quad (50)$$

We obtain (47) from (37) by the same argument leading from (48) to (49). \blacksquare

We now use assumption **A1**, that ω is a factor state.

Lemma III.2 *Let $A, B \in \mathcal{A}_S$. Then*

$$F_{A,B}(\vec{x}) := \left(\omega(A\sigma_{\vec{x}}(B)) - \omega(A)\omega(B) \right) \xrightarrow{|\vec{x}| \rightarrow \infty} 0. \quad (51)$$

Proof This follows from **A1**, **A2** and (4) of (iv), see [18], Theorem 3.2.2. \blacksquare

We are now able to state and prove our main result. Assume, with (51), that

$$F_{A,B}(\vec{x}) = O(|\vec{x}|^{-\delta}) \quad (52)$$

as $|\vec{x}| \rightarrow \infty$. Our thermal Goldstone theorem relates the rate of clustering δ in (52) to *ssb*:

Theorem III.3 (Thermal Goldstone theorem). *Let a relativistic quantum field theory be defined as a C^* -algebraic dynamical system $(\mathcal{A}_S, \omega, \tau)$, satisfying (i)–(iv) as well as **A1**–**A8**. Then, if $s \geq 3$ and if there is *ssb* as defined by (23), the rate of clustering δ in (52) must satisfy*

$$\delta \leq s - 2. \quad (53)$$

Proof By Lemma II.5, (23) implies (33), (34), which, by (45) and Lemma III.1 leads to the inequality, for any $R_0 \in \mathbb{R}$:

$$\begin{aligned} & \frac{1}{\beta} \left| \omega \left(\left[I_h(f_d \otimes g_{\tilde{R}_0}), A_{R_0} \right] \right) \right|^2 \\ & \leq i\omega(A_{R_0}^2) \omega \left(\left[I_h(f_d \otimes g_{\tilde{R}_0}), \int dt h'(t) J_0(f_d(\cdot - t)) \otimes g_{\tilde{R}_0} \right] \right). \end{aligned} \quad (54)$$

By **A6** (local current conservation)

$$\begin{aligned} & \left(\int dt h'(t) J_0(f_d(\cdot - t)) \otimes g_{\tilde{R}_0} \right) \\ & + \int dt h(t) \vec{J}(f_d(\cdot - t)) \otimes \nabla g_{\tilde{R}_0} \Omega_\omega = 0. \end{aligned} \quad (55)$$

In (55) we applied **A6** to the function $f = (h \star f_d) \otimes g_{R_0}$, where the asterisk denotes convolution.

Inserting (55) into (54) we are led to find a bound to the quantity

$$\begin{aligned} M := i\omega \left(\left[\int dt_1 h(t_1) \vec{J}(f_d(\cdot - t_1)) \otimes g_{\tilde{R}_0}, \right. \right. \\ \left. \left. \int dt_2 h(t_2) \vec{J}(f_d(\cdot - t_2)) \otimes \nabla g_{\tilde{R}_0} \right] \right). \end{aligned} \quad (56)$$

By (12) and (17), (18),

$$\frac{\partial g_{\tilde{R}_0}}{\partial x_i} = \frac{1}{\tilde{R}_0} \frac{\partial g}{\partial x_i} \left(\frac{\vec{x}}{\tilde{R}_0} \right) \quad i = 1, 2, \dots, s, \quad (57)$$

where

$$\left(\frac{\partial g}{\partial x_i} \right) \left(\frac{\vec{x}}{\tilde{R}_0} \right) = 0 \quad \text{if } |\vec{x}| \leq \tilde{R}_0 \text{ and } |\vec{x}| > \tilde{R}_0 + \delta \quad (58)$$

and thus

$$\text{supp} \left(\frac{\partial g}{\partial x_i} \right) \left(\frac{\vec{x}}{\tilde{R}_0} \right) \subseteq \Gamma_{\tilde{R}_0} \quad (59)$$

where, by (58),

$$\Gamma_{\tilde{R}_0} = \{ \vec{x} \in \mathbb{R}^s \mid \tilde{R}_0 \leq |\vec{x}| \leq \tilde{R}_0 + \delta \} \quad (60)$$

is a region of volume

$$|\Gamma_{\tilde{R}_0}| = |S_S| [(R_0 + \delta)^s - R_0^s] = O(\tilde{R}_0^{s-1}) \quad (61)$$

with $|S_S|$ the volume of a s -dimensional sphere of unit radius. Let $\Gamma_{\tilde{R}_0}^{int}$ denote the interior of $\Gamma_{\tilde{R}_0}$. We consider the cover (see Theorem VI.1 of Appendix B):

$$\Gamma_{\tilde{R}_0}^{int} = \bigcup_{i \in I} G_i \quad (62)$$

where

$$|I| = O(\tilde{R}_0^{s-1}) \quad (63)$$

and G_i are open hypercubes of side $(1 + \epsilon)$, $0 < \epsilon < 1$, in \mathbb{R}^s , there being only $O(1)$ such hypercubes along a radius, in accordance to (59): in (63), $|I|$ is the cardinality of the set I . This is of course, only one possible choice for the cover (62). In correspondence to the latter, we write now the second term in (56) following the theorem on the partition of unity in Appendix B: let

$$\beta_i = \sum_{j \in J} \alpha_j, \quad \text{supp } \alpha_j \in G_i, \quad (64)$$

corresponding to **B1**; by **B2** and **B3**, $0 \leq \beta_i \leq 1$ for all i . We define

$$i \in I \mapsto \gamma_i^r := \beta_i \frac{\partial g(\vec{x}/\tilde{R}_0)}{\partial x_r}. \quad (65)$$

Then, by (57)–(65):

$$\begin{aligned} & \int dt_2 h(t_2) \tilde{J}(f_d(\cdot - t_2) \otimes \nabla g_{\tilde{R}_0}) \\ &= \frac{1}{R_0} \sum_{r=1}^s \sum_{i \in I} \int dt_2 h(t_2) J_r(f_d(\cdot - t_2) \otimes \gamma_i^r). \end{aligned} \quad (66)$$

By locality **A7b** together with (66), we have, for M defined by (56):

$$\begin{aligned} M &= \frac{i}{R_0} \sum_{r=1}^s \sum_{i \in I} \omega \left(\left[\int dt_1 h(t_1) J_0(f_d(\cdot - t_1) \otimes g_{R_i}), \right. \right. \\ & \quad \left. \left. \int dt_2 h(t_2) J_r(f_d(\cdot - t_2) \otimes \gamma_i^r) \right] \right), \end{aligned} \quad (67)$$

where

$$R_i = O(1) \quad \forall i \in I, \quad (68)$$

is the minimal length such that $\text{supp} f_d(\cdot - t_1) \otimes g_{R_i}$ is time-like to $\text{supp} f_d(\cdot - t_2) \otimes \gamma_i^r$: it depends only on d , the support of h and the diameter of the support of γ_i^r , which is of order one by our choice of G_i in (62). By (63), (67), (68) and Assumption **A4**

$$0 \leq M \leq \text{const.} \cdot \tilde{R}_0^{s-2}, \quad (69)$$

where the constant is independent of \tilde{R}_0 . By (24), (35), (36) and the KMS condition (Assumption **A2**)

$$\begin{aligned} \omega(A_{R_0}^2) &= \frac{1}{|L_{R_0}|^2} \int_{L_{R_0}} d^3 \vec{x} \int_{L_{R_0}} d^3 \vec{y} \\ & \quad \times \left(\omega(A_0 \sigma_{\vec{x}-\vec{y}}(A_0)) - \omega(A_0)^2 \right) \\ &\leq \frac{c}{R_0^s} (2R_0)^{s-\delta} = c R_0^{-\delta}. \end{aligned} \quad (70)$$

Inserting (54), (56), (69) and (70) into (33) of Lemma II.5, we obtain, with (34):

$$0 \neq c \leq d \cdot R_0^{s-2-\delta}, \quad (71)$$

where d is a positive constant independent of R_0 . (71) is true for any $R_0 \in \mathbb{R}$; taking $R_0 \rightarrow \infty$, we obtain a contradiction unless (53) holds. \blacksquare

We now remark on the restriction to $s \geq 3$ in Theorem III.3. There are no finite-temperature equilibrium two-point functions (with vanishing chemical potential) for the massless free field for $s = 1$ and $s = 2$ and nothing is known for interacting theories (see, e.g., [34], pp. 144 and 151 for a pedagogic discussion). The proof of Theorem III.3 does not work for $s = 1$ (the surface degenerates to a point) and does work for $s = 2$ but the result is inconclusive, although (53) correctly predicts a borderline behaviour of the case $s = 2$.

For the scalar free field of mass m , the two point function corresponding to $F_{A,B}$ in (51) is, for $s = 3$:

$$\begin{aligned} W_\beta(x, m) &= (2\pi)^{-3} \int d^4 p \, \epsilon(p_0) \delta(p^2 - m^2) \\ & \quad \times (1 - e^{-\beta p_0})^{-1} e^{-i p x} \\ &= (2\pi)^{-3} \int \frac{d^3 \vec{p}}{2\omega_{\vec{p}}} e^{i \vec{p} \vec{x}} \left(\frac{e^{-i \omega_{\vec{p}} x_0}}{1 - e^{-\beta \omega_{\vec{p}}}} + \frac{e^{i \omega_{\vec{p}} x_0}}{e^{\beta \omega_{\vec{p}}} - 1} \right) \end{aligned} \quad (72)$$

where

$$\omega_{\vec{p}} := (\vec{p}^2 + m^2)^{1/2}. \quad (73)$$

For $m = 0$ the asymptotic behaviour of $W_\beta(x, m)$ for $|x_0| \ll |\vec{x}|$ is seen from (72) to be the same as that of

$$\int \frac{d^3 \vec{p}}{|\vec{p}|^2} e^{i \vec{p} \vec{x}} \cong \frac{1}{|\vec{x}|}, \quad (74)$$

which contrasts with the $\frac{1}{|\vec{x}|^2}$ fall-off in the massless $T = 0$ case. (74) is also the asymptotic rate of fall-off of $F_{A,B}(\vec{x})$ in (51) in the free massless case, and thus the result of Theorem III.3 may also be expected to be optimal in thermal (relativistic) quantum field theory, as it is in non-relativistic quantum statistical mechanics (see [5, 7]).

We conclude this section with some results and conjectures related to Theorem III.3, which help to clarify its significance. The conjectured optimality of Theorem III.3 suggests the more precise:

Conjecture III.4

(i) in the massless case ($m = 0$)

$$O(|\vec{x}|^{-\delta}) \quad \text{with } \delta \leq 1; \quad (75)$$

(ii) in the massive case ($m > 0$)

$$O(|\vec{x}|^{-\delta}) \quad \text{with } \delta > 1. \quad (76)$$

Corollary III.5 *Under Conjecture III.4, ssb of a continuous internal symmetry in thermal relativistic quantum field theory with a conserved local current implies the existence of zero mass particles in the theory.*

Thus, under Conjecture III.4 the statement of Goldstone's theorem for $T > 0$ is the same as the corresponding one for $T = 0$ (see [10][11]).

Remark III.1 *A certain form of slow decay in space-like directions has also been proved in [9] to be necessary for the existence of ssb at $T > 0$ (see (18) of [9]).*

Remark III.2 *Note that (76) does not assume exponential decay in the $m > 0$ case — as happens in the free field case (72). This is in agreement with the conjectured behaviour of the damping form factors in [24] (see also the discussion in [1]).*

Is there a spectral theoretic statement related to (75) and (76)? Since the spectrum $\sigma(L_\omega)$ of L_ω is the whole real line,

$$\sigma(L_\omega) = \mathbb{R}, \quad (77)$$

this question has no obvious answer. However, if Ω_ω is the unique (up to a phase) normalised eigenvector of L_ω with eigenvalue 0, then ω is a factor state [25] and one has the following result (see [1, 30]):

Theorem III.6 *Let Ω_ω be as above, and P^+ denote the projection onto the strictly positive part of $\sigma(L_\omega)$. Assume there exist positive constants $\delta > 0$ and $C_1(\mathcal{O})$ such that*

$$\|e^{-\lambda L_\omega} P^+ \pi_\omega(A) \Omega_\omega\| \leq C_1(\mathcal{O}) \lambda^{-\delta} \|A\| \quad (78)$$

for all $A \in \mathcal{A}(\mathcal{O})$. Consider now two space-like separated space-time regions $\mathcal{O}_1, \mathcal{O}_2$, which can be embedded into \mathcal{O} by translation and such that $\mathcal{O}_1 + re \subset \mathcal{O}_2'$, $r \gg \beta$; then, for all $A \in \mathcal{A}(\mathcal{O}_1)$, and all $B \in \mathcal{A}(\mathcal{O}_2)$

$$|\omega(BA) - \omega(B)\omega(A)| \leq C_2 r^{-2\delta} \|A\| \|B\|. \quad (79)$$

The constant $C_2(\beta, \mathcal{O}) \in \mathbb{R}^+$ may depend on the temperature $T = \beta^{-1}$ and the size of the region \mathcal{O} , but is independent of r, A , and B .

As remarked in [1], from explicit calculations one expects that $\delta = 1/2$ for free massless bosons in 3+1 space-time dimensions, and thus the exponent on the r.h.s. of (79) is optimal due to (74).

It is interesting that, in the massive case, for $T = 0$, exponential decay on the r.h.s. of (79) follows from the spectral gap in $H_\omega > 0$, i.e., exponential decay in λ of

$$\|e^{-\lambda H_\omega} \pi_\omega(A) \Omega_\omega\|, \quad (80)$$

by the cluster theorem [26], while, for $T > 0$, sufficiently fast polynomial decay of correlations — (79), with $\delta > 1/2$ — equally follows from sufficiently fast decay of

$$\|e^{-\lambda L_\omega} P^+ \pi_\omega(A) \Omega_\omega\|, \quad (81)$$

— (78) with $\delta > 1/2$ — if (76) is correct. It is to be remarked that (79) is related (see [29]) to the Buchholz-Wichmann nuclearity property [28].

IV. DISCUSSION AND OUTLOOK

In this paper we have shown that a Goldstone theorem may be proved in thermal quantum field theory, relating ssb to the space-like decay of the two-point function (Theorem III.3 of Section III). Since the limiting behaviour (53) of Theorem III.1 agrees with that of the massless free field theory (74), we were led to the conjecture that the theorem may be optimal, as occurs in non-relativistic quantum statistical mechanics, leading to a sharp distinction (75), (76) between massive and massless thermal rqft. The latter is found by examining the rate of fall-off of the two-point function only in space-like directions. If this conjecture is correct, Corollary III.5 provides a statement of Goldstone's theorem for $T > 0$, which is quite analogous to the one for $T = 0$ (see [10, 11]).

We have chosen to set our scale large only as far as space-like distances are concerned. As remarked in [27], this may be appropriate for discussing global issues like superselection sectors, statistics and symmetries. But there remains scattering theory with the associated notions of particles and inparticles, and there large time-like distances are crucial. Thus, if one is really concerned with unravelling the concept of particle in thermal rqft, the approach of Bros and Buchholz ([9, 23]) is the most natural one. However, time-like decay as $|t|^{-3/2}$ for $\vec{x} = \vec{v}t$ (which follows from (72)) leads, together with the assumption of a sharp dispersion law, to the famous Narnhofer-Requardt-Thirring theorem [30], according to which there is no interaction. We refer to [24] (see Sect. d., p. 518) for a lucid discussion of possible ways out of this dilemma, but the matter still remains under discussion.

A relevant open problem is a purely algebraic version of the Goldstone theorem in the case of positive temperature, in analogy to what was accomplished in ref. [36] for the ground state. It should also be remarked that domain problems such as the one pertaining to assumption A4 have been solved in [36], without the need of any assumption, in a very ingenious way (see (3.6) et seq), but we were unable to do the same here. In addition, nonconserved currents, successfully dealt with in [36], remains an open problem for $T > 0$. Finally, ssb of Lorentz and Galilei symmetries has been studied by a different method in ref. [37], where references to related work by Requardt are to be found.

A different but fundamental set of issues related to time-like clustering, not mentioned in [27], concerns stability. The time-like cluster property (also called mixing property [25])

$$\lim_{t \rightarrow \infty} (\omega(A\tau_t(B)) - \omega(A)\omega(B)) = 0 \quad (82)$$

implies, for $T > 0$, the dynamic stability condition of Haag, Kastler and Trych-Pohlmeyer [31]

$$\lim_{T \rightarrow \infty} \int_{-T}^T dt \omega([A, \tau_t(B)]) = 0 \quad (83)$$

(see [17], Vol. 2, Theorem 5.4.12, pg. 165). Although (82) has been proved for the ground state of relativistic quantum field theories [32], it is still open for thermal KMS states, although a similar property has been proved for a weakly dense set of (in general non-KMS) states [33]. Proof of (82) for KMS states would imply the property of return to equilibrium [25], as well as the dynamic stability condition (83), both quite deep, and in general, hard to prove (see [25] for references).

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V. APPENDIX A

Theorem V.1 *Let \mathcal{A} be a C^* -algebra and ω a state on \mathcal{A} satisfying the KMS condition (5) w.r.t. a group of norm-continuous automorphisms $\{\tau_t\}_{t \in \mathbb{R}}$. Let $A \in \mathcal{A}$ and $C \in \mathcal{A}$ be both of the form*

$$C = \int dt h(t) \tau_t(B) \quad (84)$$

with some $B \in \mathcal{A}$ and

$$\hat{h} \in \mathcal{D} = C_0^\infty(\mathbb{R}), \quad (85)$$

where \hat{h} denotes Fourier transform of h . Then

$$\frac{2}{\beta} |\omega([C, A^*])|^2 \leq \omega \left(\left[C, i \left(\frac{d}{dt} \tau_t(C) \right) \Big|_{t=0} \right] \right) \cdot \omega(\{A, A^*\}), \quad (86)$$

where $\{A, B\} := AB + BA$.

Inequality (86) (Bogoliubov's inequality) may be extended to all $A \in \mathcal{A}$ and to those C of the form

$$C = \int dt g(t) \tau_t(B), \quad (87)$$

where $g \in C^\infty(\mathbb{R})$ is such that, given any $\epsilon > 0$, there exists h satisfying (85) such that

$$\int dt |h'(t) - g'(t)| < \epsilon \quad (88)$$

and

$$\int dt |h(t) - g(t)| < \epsilon. \quad (89)$$

Proof See [14] and [17], Vol. II, pg. 333. Norm-continuity of the time-translation automorphisms was not explicitly stated in [14], but is used to extend the

result from A in the class (84), (85) to the whole of \mathcal{A} by density (see [20], Theorem 4.8). The extension to (87) was not mentioned in [14], but follows from (84), (85), (86), (87) and (88), together with

$$\frac{d}{dt} \tau_t(C) \Big|_{t=0} = \int dt g'(t) \tau_t(B) \quad (90)$$

for C of the form (87), and

$$\frac{d}{dt} \tau_t(C) \Big|_{t=0} = \int dt h'(t) \tau_t(B) \quad (91)$$

for C of the form (84). ■

Corollary V.2 *Let $A \in \mathcal{A}_S$ (see Section II) and C be of the form (87), with $B \in \mathcal{A}_S$ and*

$$g \in C_0^\infty(\mathbb{R}). \quad (92)$$

Then, if ω is a state on \mathcal{A}_S satisfying the KMS condition, Condition A3 holds.

Proof Since $h \in \mathcal{S}(\mathbb{R})$ by (85), given g satisfying (92), we may choose the ‘tail to infinity’ in h appropriately so that (88) and (89) hold. ■

Remark V.1 *We use (86) in the main text under conditions of Corollary V.2. Since conditions (85) and (92) are mutually excludent by the Paley-Wiener theorem (see e.g. [21], Exercise 8 of Chap. 10), the density argument in Theorem V.1 is important for the application we make of (86) in Section III.*

Remark V.2 *For the proof of positivity of the middle term in (85) and other questions related to the Bogoliubov scalar product, see ([17], Vol. II, pg. 334). For some inequalities in statistical mechanics for W^* -systems, see [22].*

VI. APPENDIX B

We state here, for the reader's convenience, the theorem (partition of unity) used in Theorem III.3 of Section III:

Theorem VI.1 (see [23], Theorem, p. 61, Chap. I, Sect. 12). *Let G be an open set of \mathbb{R}^n , and let a family of open sets $\{G_i \mid i \in I\}$ cover G , i.e., $G = \bigcup_{i \in I} G_i$. Then there exists a system of functions $\{\alpha_j(x) \mid j \in J\}$ of $C_0^\infty(\mathbb{R}^n)$ such that*

B1 *for every $j \in J$, $\text{supp}(\alpha_j)$ is contained in some G_i ;*

B2 *for every $j \in J$, the function α_j satisfies $0 \leq \alpha_j(x) \leq 1$ for all $x \in \mathbb{R}^n$;*

B3 $\sum_{j \in J} \alpha_j = 1$ *for $x \in G$.*

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